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THE MATHEMATICAL MONTHLY.

AUGUST, 1859.

EDITED BY

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CONTENTS.

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THE

MATHEMATICAL MONTHLY.

Vol. I...AUGUST, 1859....No. XI.

PRIZE PROBLEMS FOR STUDENTS.

I.

Solve the equations

$$x+y=a$$
$$(x^3+y^3)(x^2+y^2)=b,$$

and give a discussion of the values of the roots.

II.

Let A, B, C be the angles, and a, b, c the opposite sides, of a plane triangle; it is required from the relation

$$\frac{\sin A}{a}=\frac{\sin B}{b}=\frac{\sin C}{c}$$

to deduce the formula

$$a^2=b^2+c^2-2\,b\,c\cos A.$$

III.

A number n of equal circles touch each other externally, and include an area of a square feet; to find the radii of the circles. (Communicated by Artemas Martin, Esq.)

IV.

If the sides of a spherical trapezium be denoted by a, b, c, d, the

diagonals by δ_1 and δ_2, and the distance between the middle points of the diagonal by Δ; show that

$$\cos a + \cos b + \cos c + \cos d = 4 \cos \tfrac{1}{2} \delta_1 \cos \tfrac{1}{2} \delta_2 \cos \Delta.$$

(Communicated by GEORGE EASTWOOD, Esq.)

V.

From an urn containing four white and four black balls, four are repeatedly drawn and replaced. *A* agrees to pay *B* one dollar every time the four balls drawn are equally divided between white and black; but if three, or all four, are of the same color, *B* is to pay *A* one dollar. Who has the advantage, and what is its value for each drawing? (Communicated by SIMON NEWCOMB, Esq.)

The solution of these problems must be received by the first of October, 1859.

REPORT OF THE JUDGES UPON THE SOLUTIONS OF THE PRIZE PROBLEMS IN No. VII., Vol. I.

THE first Prize is awarded to O. B. WHEELER, student in the University of Michigan, at Ann Arbor.

The second Prize is awarded to WILLIAM EGERTON, student in the Baltimore College, Baltimore, Maryland.

PRIZE SOLUTION OF PROBLEM I.

"On two sides AC and BC of any triangle ABC, let any parallelograms $ACDE$ and $BCFG$ be described. Let ED and FG produced, meet in H; join HC, and through A and B draw AL and BM equal and parallel to HC. Join LM. It is required to prove that the parallelogram $ALMB$ on the side AB is equal to the sum of the parallelograms on the sides AC and BC.

"Show also that the Pythagorean proposition is a particular case of this proposition."— PAPPUS.

The parallelograms AH and AD are equivalent; so are the parallelograms BH and BF, since they have the same base and altitude. Therefore $LH = AC$, and $HM = CB$. But since AL and BM are each equal and parallel to HC, they are equal and parallel to each other, and $LM = AB$. Therefore the triangle LHM is equal to ABC, since they are mutually equilateral. Take away the triangle OCP from each, and we have left $LHMPCO$, equivalent to $AOPB$. Add ALO and BMP to both, and we have

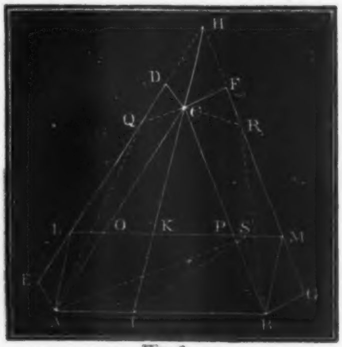


Fig. 1.

$$ALHC + BMHC = AEDC + BGFC = ALMB.$$

Or thus: Produce HC till it meets AB in I; then will $AD = AH = AK$. For $AI = LK$, and the triangles $LHK = ACI$. Taking away the common part OCK, and adding ALO to both remainders, we have $AH = AD = AK$. Also $BF = BK$. $\therefore AD + BF = AM$.

Corollary. The parallelograms AK and BK have the same altitude, and are therefore to each other as their bases AI and BI. Hence $AD : BF :: AI : IB$; that is, the line HC produced cuts the third side AB into parts which are to each other as the parallelograms described on the adjacent sides.

Cor. 2. If triangles be described on the sides AC and BC, and lines be drawn through their vertices Q and R parallel to the sides AC and BC, and produced till they meet in H, and the remaining lines be drawn as in the figure, then the vertex of the triangle described on AB, which shall be equivalent to the sum of the other two, will be found anywhere in the line LM; for a triangle is equivalent to half the parallelogram having the same base and altitude.

COR. 3. The triangles AQC and BRC are to each other as $AI:BI$, since the triangles are the halves of AD and BF.

COR. 4. The triangle ASI is equivalent to AQC, and BSI to BRC.

COR. 5. When ACB is right-angled at C, and squares are described on AC and BC, the diagonal HC equals AB. For the two triangles CDH and ACB have $AC = CD$, $BC = DH$, and C and D right angles. $\therefore CH = AB = AL$. Also the angle $DCH = CAB$, and $DHC = CBA = HCF = LAC$. $\therefore DCH + DHC = CAB + LAC$ = a right angle. $ALMB$ is therefore a square equivalent to the sum of the other two.

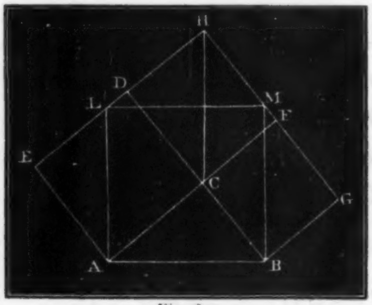


Fig. 2.

This solution is by DAVID TROWBRIDGE. In most of the other solutions, HC (Fig. 1) was produced to I; then $AD = AH = AK$, and $BF = BH = BK$, as is easily seen. Therefore, by addition, $AD + BF = AM$.

PRIZE SOLUTION OF PROBLEM II.

"The area of a right-angled triangle is equivalent to the rectangle of the differences between the radius of the inscribed circle and the two shorter sides respectively; or the rectangle of the segments of the hypothenuse made by a perpendicular let fall upon it from the centre of the inscribed circle."

Let b be the base and a the altitude of the triangle ABC, and r the radius of the inscribed circle. It will be readily seen that the triangle $GDA = ADE = DAH$. $\therefore GDEA = EDHA$. $GDC = CDF = DCI$. $\therefore GDFC = FDIC$.

$$ACB = CIDHAB = ar + (b - r)r$$
$$= ar + br - r^2.$$

$$KCA = ACB = ab - (ar + br - r^2) = ab - ar - br + r^2;$$

but $\quad KIDH = (a - r)(b - r) = ab - ar - br + r^2.$

$$\therefore ABC = KIDH = (a - r)(b - r) = AE \times CF = AG \times CG.$$

This solution is by C. Herschel.

Prize Solution of Problem III.

"The distance between two points, A and B, is a miles. A person starts at A and travels the first day one mth his distance to B; the second day he travels back one mth his distance to A; the third day he turns and travels one mth his distance to B, and so on. How far will he travel in n days, and how far will he be from A?"

Let a, a_1, a_2, &c., denote his distances from the point A or B towards which he travels on the successive days; and let x_1, x_2, x_3, &c., denote the distances travelled. Then, by the question

$$(1) \qquad \frac{a}{m} = x_1, \ \frac{a_1}{m} = x_2, \ \frac{a_2}{m} = x_3, \text{ \&c.,}$$

and by addition

$$(2) \qquad \frac{1}{m}(a + a_1 + a_2 + \dots a_{n-1}) = x_1 + x_2 + x_3 + \dots x_n = S_n$$

the whole distance travelled in n days. Next we have

$$(3) \qquad \begin{array}{lll} a = a, & m x_1 = m x_1, & x_1 = x_1 \\ a_1 = a - a + x_1, & m x_2 = m x_1 - m x + x_1, & x_2 = x_1 - \frac{m-1}{m} x_1, \\ a_2 = a - a_1 + x_2, & m x_3 = m x_1 - m x_2 + x_2, & x_3 = x_1 - \frac{m-1}{m} x_2, \\ \vdots \quad \vdots \quad \vdots & \vdots \quad \vdots \quad \vdots & \vdots \quad \vdots \quad \vdots \\ a_n = a - a_{n-1} + x_n, & m x_{n+1} = m x_1 - m x_n + x_n, & x_{n+1} = x_1 - \frac{m-1}{m} x_n. \end{array}$$

By addition, the third set gives

$$S_n + x_{n+1} = (n + 1) x_1 - \frac{m-1}{m} S_n;$$

hence, and by (1)

$$(4) \qquad S_n = \frac{(n+1) m x_1 - m x_{n+1}}{2m - 1} = \frac{(n+1) a - a_n}{2m - 1}.$$

To find the value of $a_n = m x_{n+1}$, put $\frac{m-1}{m} = p$, and by succes-

sive substitutions, the last set of equations (3) give

$$(5)\quad \begin{array}{l} x_1 = x_1,\ x_2 = x_1(1-p),\ x_3 = x_1(1-p+p^2), \text{ and so on to} \\ x_{n+1} = x_1(1-p+p^2-p^3+p^4-\ldots\pm p^n). \end{array}$$

The upper sign is to be used when n is *even*, and the lower when it is *odd*. Summing the series (5) becomes

$$(6)\quad x_{n+1} = x_1\frac{1\pm p^{n+1}}{1+p}, \text{ and } m\,x_{n+1} = a_n = a\frac{1\pm p^{n+1}}{1+p};$$

or, replacing the value of p,

$$(7)\qquad a_n = m\,x_{1+n} = a\,\frac{m^{n+1}\pm(m-1)^{n+1}}{m^n(2m-1)}.$$

Substituting this value of a_n in (4) and reducing, we get

$$(8)\qquad S_n = \frac{(n+1)\,a}{2m-1} - \frac{m^{1+n}\pm(m-1)^{n+1}}{m^n(2m-1)^2}.$$

When n is odd, (7) denotes the distance from A; but when n is even, this distance is $a - a_n$. This solution is by DAVID TROWBRIDGE.

PRIZE SOLUTION OF PROBLEM IV.

"The volume of any right cone equals the product of its whole surface by one third the radius of the inscribed sphere." — Communicated by Prof. SNELL.

Let a sphere be inscribed in the right cone. Circumscribe a regular polygon about the base of the cone, and join the vertices of the polygon with the vertex of the cone. The faces of this circumscribed pyramid will be tangents to both the cone and sphere. Join the vertices of the solid angles of the pyramid and the centre of the sphere, and thus divide it into as many other pyramids as it has faces. All these pyramids have a common vertex at the centre of the sphere, and their common altitude is the radius of the sphere, since all their bases are tangents to it. The solidity of each pyramid is the product of its base by one third of the radius of the sphere, and therefore the solidity of the pyramid circumscribing the cone is the product of its whole surface by one third of the radius of the sphere. Let, now,

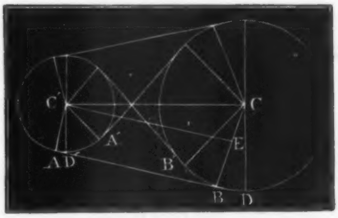
the number of sides of the polygon circumscribing the cone's base be continually increased; the limit of the polygon is the base of the cone; and the limit of the whole surface of the pyramid is the whole surface of the cone. Hence the volume of the cone is the product of its whole surface by one third of the radius of the inscribed sphere.

This solution is by J. C. ELLIOTT; and the same reasoning was used by O. B. WHEELER.

PRIZE SOLUTION OF PROBLEM V.

"One pulley drives another by means of a belt; give the length of the belt l, the diameter D, of the larger pulley, the distance a between the centres of the pulleys; to find the diameter d of the smaller pulley. Find also a simple approximate formula for the use of machinists."

Let $CB = R = \frac{1}{2} D$, and $C' A = r = \frac{1}{2} d$. If b, D, and a remain the same, it is evident that d will have different values according as the belt is a crossed or open one. For a crossed belt the sum of the straight parts is $2\sqrt{a^2 - (R+r)^2}$, since each is a leg of a right-triangle, having a for the hypothenuse and $R + r$ for the other leg.



In this triangle the angle opposite $R + r$ equals $B' C D$ or $A C' A'$, or $\sin^{-1} \frac{R+r}{a}$, and therefore the arc $D B B'$ equals $R \sin^{-1} \frac{R+r}{a}$. The curved part of the belt on the greater pulley is then $R\left(\pi + 2\sin^{-1}\frac{R+r}{a}\right)$, and on the smaller pulley it is $r\left(\pi + 2\sin^{-1}\frac{R+r}{a}\right)$.

$$\therefore\ l = 2\sqrt{a^2 - (R+r)^2} + (R+r)\left(\pi + 2\sin^{-1}\frac{R+r}{a}\right).$$

From this equation r can be found approximately. For an open belt, $A B$ or $C' E$ is $\sqrt{a^2 - (R - r^2)}$. The angle $B C D = C C' E$, the

sine of which is $\frac{R-r}{a}$. The arc BD equals $R\ \sin^{-1}\frac{R-r}{a}$, and the arc AD' equals $r\sin^{-1}\frac{R-r}{a}$.

$$\therefore\ l = 2\sqrt{a^2-(R-r)^2} + R\left(\pi + 2\sin^{-1}\frac{R-r}{a}\right) + r\left(\pi - 2\sin^{-1}\frac{R-r}{a}\right)$$
$$= 2\sqrt{a^2-(R-r)^2} + (R+r)\,\pi + 2\,(R-r)\sin^{-1}\frac{R-r}{a}.$$

From this equation another approximate value of r can be found.

Since $R-r$ is small compared with a, we have approximately,

$$2\sqrt{a^2-(R-r)^2} = 2\,a - \frac{(R-r)^2}{a},\quad \sin^{-1}\frac{R-r}{a} = \frac{R-r}{a}.$$

Therefore, for an open belt the approximate formula is

$$= 2\,a - \frac{(R-r)^2}{a} + (R+r)\,\pi + \frac{2\,(R-r)^2}{a},$$
$$= 2\,a + \frac{(R-r)^2}{a} + (R+r)\,\pi,$$

from which

$$2\,r = d = 2\,R - a\,\pi \pm \sqrt{8\,(a^2+r^2) - l\,a + a^2\pi^2}.$$

This solution is by O. B. Wheeler.

Solution 2d. If the angle $C\,C'\,E = B\,C\,D$ be denoted by φ, then $2\,AB = 2\,a\cos\varphi$, $BE = R - r = a\sin\varphi$, $BD = R\,\varphi$, $AD' = r\,\varphi$.

$$\therefore\ l = 2\,a\cos\varphi + R\,\pi + 2\,R\,\varphi + r\,\pi - 2\,r\,\varphi$$
$$= 2\,a\cos\varphi + (R+r)\,\pi + 2\,(R-r)\,\varphi$$
$$= 2\,a\cos\varphi + (R+r)\,\pi + 2\,a\,\varphi\sin\varphi$$
$$(1)\qquad = 2\,a\cos\varphi + 2\,a\,\varphi\sin\varphi + 2\,R\,\pi - a\,\pi\sin\varphi.$$

But
$$\cos\varphi = 1 - \tfrac{1}{2}\varphi^2 + \tfrac{1}{1\cdot2\cdot3\cdot4}\varphi^4 - \&c.,$$
$$\sin\varphi = \varphi - \tfrac{1}{1\cdot2\cdot3}\varphi^3 + \tfrac{1}{1\cdot2\cdot3\cdot4\cdot5}\varphi^5 - \&c.;$$

and substituting these values of $\cos\varphi$ and $\sin\varphi$ in (1) we shall find, after reversing the series, that

$$\varphi = -\frac{2}{\pi}\left(\frac{l-2\,R\,\pi-2\,a}{2\,a}\right) + \frac{4}{\pi^3}\left(\frac{l-2\,R\,\pi-2\,a}{2\,a}\right)^2 - \&c.$$

But $\quad 2r = d = 2R - 2a\sin\varphi$

$$= D - 2a\left(\varphi - \tfrac{1}{1\cdot2\cdot3}\varphi^3 + \tfrac{1}{1\cdot2\cdot3\cdot4\cdot5}\varphi^5 - \&c.\right).$$

For a simple approximate formula, let $\cos\varphi = 1$, and $\sin\varphi = \varphi$, and (1) becomes

$$l = 2a + 2a\varphi^2 + 2R\pi - a\pi\varphi,$$

$$\therefore \quad \varphi = \tfrac{1}{4}\pi \pm \sqrt{\tfrac{1}{16}\pi^2 + \frac{l - 2R\pi - 2a}{2a}}.$$

$$\therefore \quad d = D - 2\left(\tfrac{1}{4}\pi \pm \sqrt{\tfrac{1}{16}\pi^2 + \frac{l - 2R\pi - 2a}{2a}}\right).$$

This solution is by Asher B. Evans.

Joseph Winlock.
Chauncey Wright.
Truman Henry Safford.

NOTES AND QUERIES.

1. *Subtraction of Fractions.* Let $\frac{a}{b}$ and $\frac{c}{d}$ be the fractions; then, as is easily seen,

$$\frac{a}{b} - \frac{c}{d} = \frac{a(d-c) - c(b-a)}{bd} = \frac{ad - bc}{bd}.$$

When $d-c$ is less than d, and $b-a$ is less than b, the numerator of the second form will be most readily computed; and the advantage will be great when the terms of the fractions are large numbers, but nearly equal to each other. Terquem's *Annales de Mathematique.*

2. It is interesting to notice, that the demonstration of the Pythagorean proposition on page 231 of the Mathematical Monthly is essentially the same as the Indian demonstration contained in the *Bija Ganita*, and referred to as the "figure of the bride's chair," &c. It is involved in the square AK (fig. p. 231). For let a represent the side of the square, and b and c the legs of the equal

right-angled triangles. Then the area of the four equal triangles is $2bc$; and $(b-c)^2$ is the area of the square SQ. *Whence $a^2 = 2bc + (b-c)^2 = b^2 + c^2$. This demonstration is given by Dr. HUTTON (Tracts, London, 1812, 3 Vols. 8vo), in his History of Algebra, where may be found a very complete description of the *Bija Ganita* and *Lilawati*. Dr. HUTTON had in his possession Persian MSS. containing translations of both these works.—H. W. RICHARDSON, Waterville College, Maine.

3. *Problem.* A man left 17 horses to be divided among his three sons, the first to have $\frac{1}{2}$, the second $\frac{1}{3}$, and the youngest $\frac{1}{4}$ of the number. They could not agree as to the division, because it required some of the horses to be divided; thus, $\frac{1}{2}$ of $17 = 8\frac{1}{2}$, $\frac{1}{3}$ of $17 = 5\frac{2}{3}$, $\frac{1}{9}$ of $17 = 1\frac{8}{9}$. The sum of the three shares was $16\frac{1}{18}$ horses, and the whole number was not distributed. They carried the case to a judge, who told them, that, if they would abide by his decision, he would give each more than his share, and each should have a whole number of horses. Accordingly, he brought his own horse from the stall, and put him with the 17 others. He then gave $\frac{1}{2}$ of $18 = 9$ horses to the first; $\frac{1}{3}$ of $18 = 6$ to the second; and $\frac{1}{9}$ of $18 = 2$ to the third, making 17 in all. He then returned his own horse to the stall, and left the sons well satisfied. Was the decision just? * * *

4. A correspondent sends us the equation $2\pi\sqrt{-1} = 0$, and proves it as follows. In exponentials,

---

* This demonstration has been sent us by JOHN M. BROWN, Esq., of Frankfort, Ky. We have also received several others, which are filed for publication. Prof. ALPHEUS CROSBY, Principal of the Normal School at Salem, Mass., in his work on Geometry, refers to a *Treatise on the Pythagorean Proposition* by HOFFMAN, published at Mayence in 1819, which contains thirty-three different demonstrations of this celebrated theorem. We intend, as soon as we can get a copy of the work, to give in the MONTHLY a brief outline of each demonstration. But if any of our readers have the work, we shall be much obliged for such a synopsis as we propose.

$$\sin x = \frac{1}{2\sqrt{-1}}\left(e^{x\sqrt{-1}} - e^{-x\sqrt{-1}}\right).$$

Make $x = \pi$; then $\sin \pi = 0 = \frac{1}{2\sqrt{-1}}\left(e^{\pi\sqrt{-1}} - e^{-\pi\sqrt{-1}}\right)$; or $e^{\pi\sqrt{-1}} = e^{-\pi\sqrt{-1}}$. Multiplying both members by $e^{-\pi\sqrt{-1}}$ we get $e^{2\pi\sqrt{-1}} = e^{\circ} = 1$. $\therefore\ 2\pi\sqrt{-1} = 0$.

The last conclusion is not correct; for if it were, some one of the factors of $2\pi\sqrt{-1}$ must be zero, which is not the case.

Make $x = n\pi$; then $e^{n\pi\sqrt{-1}} = e^{-n\pi\sqrt{-1}} = \frac{1}{e^{n\pi\sqrt{-1}}}$. Therefore, $e^{2n\pi\sqrt{-1}} = 1$. $\therefore \log 1 = 2n\pi\sqrt{-1}$. And since this is true for all values of n, it follows that $\log 1 = 0$, for $n = 0$, but is imaginary for all other values of n; that is, log 1 has an infinite number of values, only one of which is real.

It is true in general, that any positive number has one real, and an infinite number of imaginary, logarithms. Let x be the Napierian logarithm of N. Then $Ne^{2n\pi\sqrt{-1}} = N = e^x$. Therefore, $\log N + 2n\pi\sqrt{-1} = x$; or $\log N = x - 2n\pi\sqrt{-1}$, which is real only when $n = 0$.

5. *The least Common Multiple.* Some of the arithmetics say, divide the given numbers by *any* number which will divide two or more of them without a remainder, next divide the quotients and undivided numbers by *any* number, &c.; others say, divide by *any prime* number. Will both rules give the correct answer?

6. What is the origin of the term, *Pons Asinorum,* as applied to the fifth proposition of the first book of Euclid?

7. *Twenty-Two Systems of Coördinates.* The usual right-line, or Cartesian coördinates, are x, y; the polar r, φ; the directions of the normal and tangent, or the angles they make with an assumed axis, are ν and τ; s is the length of the curve; ρ is its radius

of curvature; ε the angle between the radius vector and tangent.

$$(1)\, y=F(x)\,.\ (2)\, r=F(\varphi)\,.\ (3)\, \varrho=F(r)\,.\ (4)\, \tau=F(s).$$
$$(5)\, \varrho=F(s)\,.\ (6)\, \varepsilon=F(\varphi)\,.\ (7)\, \tau=F(\varphi)\,.\ (8)\, r=F(x).$$
$$(9)\, x=F(\varphi)\,.(10)\, x=F(\varepsilon)\,.(11)\, \tau=F(x)\,.(12)\, \varrho=F(x).$$
$$(13)\, x=F(s)\,.(14)\, \varepsilon=F(r)\,.(15)\, \tau=F(r)\,.(16)\, \varrho=F(r).$$
$$(17)\, r=F(s)\,.(18)\, \varrho=F(\varphi)\,.(19)\, \varphi=F(s)\,.(20)\, r=F(\varepsilon).$$

$(21)\, \varrho=F(\varepsilon)\,.(22)\, \varepsilon=F(s)$. The student will find it an excellent exercise to express some familiar curve in as many of these systems as possible. [See Paper on this subject by Rev. THOMAS HILL, in Proc. Am. As. Ad. Sc. 12th meeting, p. 1.]

ANOTHER SOLUTION OF PRIZE PROBLEM II., No. IV.

By GEORGE EASTWOOD, Saxonville, Mass.

I. IN solving this problem, I shall use Professor GUDERMANN's method of Spherical Rectangular Coördinates, on account of the remarkable analogy which it exhibits between the properties of lines drawn on the surface of a sphere, and those of lines drawn in a plane. I shall assume, as Mr. MERRILL does in his solution, (No. VIII.), that ABC is the proposed triangle, AB the given base, P its pole, PO a prime meridian passing through the middle of the base, and PCD another meridian passing through the vertex C; that the side AC intersects PO in E, and that BC meets it in F.

Let the vertex C be projected on PO in the point G, and put $AO=OB=\alpha$, $OE=\beta$, $OF=\beta'$, $OD=x$, and $OG=y$. Then, if we agree to represent the trigonometric tangents of the coördi-

nate arcs, by the symbols of those arcs the equation of the side AC will be defined by

(1) $$y = \frac{\beta}{\alpha} x + \beta,$$

and of BC by

(2) $$y = -\frac{\beta'}{\alpha} x + \beta'.$$

By spherics,

$$\tan A = \frac{\beta \sec \alpha}{\alpha},$$

(3) $$= \frac{y \sec \alpha}{\alpha + x}, \text{ by reason (1).}$$

$$\tan B = \frac{\beta' \sec \alpha}{\alpha},$$

(4) $$= \frac{y \sec \alpha}{\alpha - x}, \text{ by reason of (2).}$$

But $$\frac{(3)}{(4)} = \frac{\alpha - x}{\alpha + x}$$

is, by the question, a given ratio $= m$, suppose.

(5) $$\therefore\ x = \frac{(1-m)\alpha}{1-m};$$

that is, the vertex of the triangle is always on the meridian circle PCD.

From (5) we have, since x and α are tangents,

$$\sin x = \frac{\pm (1-m) \sin \alpha}{[(m+1)^2 \cos^2 \alpha + (1-m)^2 \sin^2 \alpha]^{\frac{1}{2}}},$$

$$\cos x = \frac{\pm (m+1) \cos \alpha}{[(m+1)^2 \cos^2 \alpha + (1-m)^2 \sin^2 \alpha]^{\frac{1}{2}}}.$$

Hence $$\sin AD = \sin(\alpha + x),$$

$$= \frac{\sin 2\alpha}{(m^2 + 2m \cos 2\alpha + 1)^{\frac{1}{2}}}.$$

If in this equation we make $2\alpha = a$, we shall have exactly the same result that Mr. MERRILL obtains in his solution, page 259.

II. By the same method of coördinates we shall find that Prize Problem II., No. V., is susceptible of a very neat and simple solution. For, if we designate the required coördinates by x and y, the given

points by $x' y'$ and $x'' y''$, and the intercepts of the axes of reference by α and β; then, by known properties* of great circles arcs, we have

$$(1) \qquad x = -\frac{1}{\alpha}, \quad y = -\frac{1}{\beta},$$

$$(2) \qquad y' = -\frac{\beta}{\alpha} x' + \beta,$$

$$(3) \qquad y'' = -\frac{\beta}{\alpha} x'' + \beta.$$

The elimination of α and β from (1), by means of (2) and (3), will obviously satisfy the required conditions of the problem. But the elegant solution of the problem by Mr. OSBORNE in the last MONTHLY, page 292, by an entirely different method, would seem to render further remarks unnecessary.

THE MOTIONS OF FLUIDS AND SOLIDS RELATIVE TO THE EARTH'S SURFACE.

[Continued from page 307.]

SECTION IV.

ON THE GENERAL MOTIONS AND PRESSURE OF THE ATMOSPHERE.

33. BY the general motions of the atmosphere are meant all those motions produced by a difference of density between the equatorial and polar regions arising principally from a difference of temperature. If the motions of the atmosphere were not resisted by the earth's surface, the results of the preceding sections could be at once applied to them without any modifications, and hence towards the poles there would be a very rapid motion eastward, and in the equatorial

* These properties, and many other analogous ones, of great circle arcs, it is proposed to investigate in subsequent numbers of the MONTHLY.

regions towards the west, and the atmosphere would entirely recede from the poles, and be also depressed about 4,000 feet at the equator, as has been shown in section (2). Although the preceding results, when applied to the atmosphere, are very much modified by the resistances of the earth's surface, yet they will be of great advantage in explaining its general motions; for as there can be no resistance until there is motion, the atmosphere must have a tendency to assume, in some measure, the same motions and figure as in the case of no resistances. Hence, towards the poles the general motions of the atmosphere must be towards the east, and in the torrid zone towards the west; but as these motions, in consequence of the resistances, are small in comparison with those in the case of no resistances, instead of the atmosphere's receding entirely from the poles, as represented in Fig. 1, page 215, there must be only a comparatively small depression there, as represented in Fig. 5, and instead of its being about 4,000 feet lower at the equator than at the place of its maximum height near the tropics (§ 18), there must be only a very slight depression there.

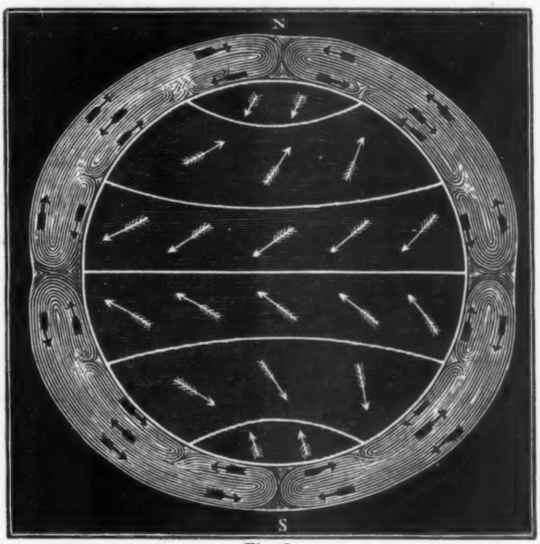


Fig. 5.

34. The force which overcomes the resistance of the earth's surface to the east and the west motions of the atmosphere depends upon the term in the least of our general equations (13) containing

$D_t\theta$ as a factor, which depends upon the interchanging motion of the fluid between the equatorial and the polar regions, and hence the term must vanish at the equator and the poles. All the east or west motion of the atmosphere is consequently destroyed by the resistances at these places, and hence as $D_t\theta$ vanishes there also, there is a belt of calms at the equator, called the equatorial calm belt, and there must be also a region of calms about the poles.

35. As the motion of the atmosphere is east towards the poles and west near the equator, somewhere between the equator and the poles there must be a parallel of no motion east or west, which, in the case of no resistance, was determined upon the hypothesis of an initial state of rest, and found to be at the parallel of 35°, § (18). In the case of the atmosphere this parallel is entirely independent of the initial state of the atmosphere, and depends in a great measure upon the law of resistance, and hence it cannot be accurately determined. It is evident, however, that the east and west motions of the atmosphere at the earth's surface must be such that the sum of the resistances of each part of the earth's surface multiplied into its distance from the axis of rotation, must be equal 0, else the velocity of the earth's rotation would be continually accelerated or retarded, which cannot arise from any mutual action between the surface of the earth and the surrounding atmosphere. Now, as the part of the earth's surface where the motion of the atmosphere is west is much farther from the axis than the part where it is east, the latter part must comprise more than half of the earth's surface, unless the velocity of the eastern motion towards the poles is much greater than that of the western motion near the equator. Therefore, since one-half of the earth's surface is contained between the parallels of 30°, the parallels of no east or west motion at the earth's surface must fall within these parallels, and they are accordingly found to be near the tropics, on the ocean. Hence the maximum height

of the atmosphere, as represented in Fig. (5), must also be near the same parallels.

36. The increase of pressure arising from the accumulation of atmosphere near the tropics, caused principally by the deflecting forces (§ 32) arising from the more rapid east and west motions of the atmosphere in the upper regions, where there is least resistance, gives the atmosphere a tendency to flow from beneath this accumulation both towards the equator and the poles, since the motions, and consequently the forces, which cause this accumulation, are much less near the surface. But on account of the greater density of the atmosphere towards the poles, it has a tendency also to flow, at the earth's surface, from the poles towards the equator. Between the parallels of greatest pressure and the equator, these tendencies combine, and produce a strong surface current, which, combining with the westward motion there, gives rise to the well-known north-east wind in the northern hemisphere, and the south-east wind in the southern hemisphere, called the trade winds. But between the parallels of greatest pressure and the poles, these tendencies are opposed to each other, and the one arising from the accumulation of atmosphere near the tropics being the greater in the middle latitudes, causes the atmosphere to flow at the earth's surface towards the poles; and this motion, combining with the general eastward motion of the atmosphere in those latitudes, gives rise to the south-west wind in the northern hemisphere and the north-west wind in the southern hemisphere, called the passage winds.

37. Near the poles, the tendency to flow towards the equator seems to be the greater, and causes a current there *from* the poles, which, being deflected westward (§ 32), causes a slight north-east wind in the north frigid zone, and a south-east wind in the south frigid zone. But this is only near the earth's surface; and the gener-

al tendency of the atmosphere in the upper regions must be towards the east, as will be seen.

38. Since the atmosphere near the tropics can have no motion in any direction at the earth's surface, there are calm belts there, called the tropical calm belts. Near the polar circles, where the polar and passage winds meet, there must also be calm belts, which may be called polar calm belts. The motions of the atmosphere, therefore, at the earth's surface, if they were not modified by the influence of continents, would be as represented in the interior of Fig. (5), in which the heavy lines represent the calm belts. On account of the influence of the continents, these belts are somewhat displaced and irregular, and on account of the varying position of the Sun, they change their positions a little in different seasons of the year.

The southern limit of the polar winds in the northern hemisphere, and also the limit between the trade and passage winds, has been determined by Prof. J. H. Coffin, from the discussion of a great number of observations at different points, and given in a chart, in his treatise on the winds, published in the seventh volume of the Smithsonian Contributions.

39. That the atmosphere is depressed at the equator and the poles, and has its maximum height near the tropics, as has been represented, is indicated by barometrical pressure. It was formerly thought that this pressure, at the level of the ocean, was very nearly 30 inches in all latitudes; but it is now well established that it is much less towards the poles than near the tropics, and also a little less at the equator. Says Captain Wilkes: "The most remarkable phenomenon which our observations have shown is the irregular outline of the atmosphere surrounding the earth as indicated by the pressure upon the measured column at different parts of the surface. Our barometrical observations show a depression

within the tropics, a bulging in the temperate zone, again undergoing a depression on advancing towards the arctic and antarctic circles." The mean of all the observations, as given in the Report of the Exploring Expedition, from Cape Henry to Madeira, taken between the parallels of 28° and 32°, was 31.215 inches; at Maderia, latitude 32° 53′, 30.176 inches; and in the rainy belt between the parallels of 8° and 12°, 29.987 inches. After passing the equator there was a slight elevation, again reaching its maximum near the tropic of Capricorn. Beyond this there was a gradual depression until about the parallel of 55°, where the barometer was rapidly depressed below 29 inches. After doubling Cape Horn and proceeding towards the equator, the height of the barometer gradually increased again to its usual height in the middle and equatorial latitudes. On sailing south again, in the Pacific Ocean, a depression of the barometer was again observed. The mean of all the observations taken on 22 days, in sailing from Callao to Tahiti, between the parallels of 10° and 15°, was 30.109 inches; and of those made on 32 days, between the parallels of 15° and 20°, was 30.147 inches. The mean of the observations made on 5 days, after leaving Sydney, between the parallels of 35° and 45°, was 30.305 inches; of those made on 7 days, between the parallels of 45° and 55°, was 29.790 inches; of those taken on 8 days, between the parallels of 55° and 65°, was 29.378 inches. The mean also of all those taken along the antarctic continent was 29.040 inches.

40. Says Sir James Ross (*Voyage to the Southern Seas*, Vol. 2, p. 383): "Our barometrical experiments appear to prove that the atmospheric pressure is considerably less at the equator than near the tropics; and to the south of the tropic of Capricorn, where it is greatest, a gradual diminution occurs as the latitude is increased, as will be shown from the following Table, derived from hourly observations of the height of the column of mer-

cury between the 20th of November, 1839, and the 31st of July, 1843."

EXTRACT FROM ROSS'S TABLE.

LATITUDE.	PRESSURE.	LATITUDE.	PRESSURE.	LATITUDE.	PRESSURE.
	inches.		inches.		inches.
Equator,	29.974	42° 53′	29.950	55° 52′	29.360
13° 0′ S.	30.016	45 0	29.664	60 0	29.114
22 17	30.085	49 8	29.467	66 0	29.078
34 48	30.023	51 33	29.497	74 0	28.928
		54 26	29.347		

41. The following table, first published by M. Schouw, and reduced here from millimetres to English inches, shows that there is a similar bulging of the atmosphere in the middle latitudes, and depression at the pole in the northern hemisphere, as has been observed in the southern hemisphere.

PLACE.	LATITUDE.	PRESSURE.	PLACE.	LATITUDE.	PRESSURE.
		inches.			inches.
Cape,	33° 0′ S.	30.040	London,	51° 30′	29.961
Rio Janeiro,	23 S.	30.073	Altona,	53 30	29.937
Christianburg,	5 30 N.	29.925	Dantzic,	54 30	29.925
La Guayra,	10	29.928	Konigsberg,	54 30	29.941
St. Thomas,	19	29.941	Apenrade,	55	29.905
Macao,	23	30.039	Edinburgh,	56	29.851
Teneriffe,	28	30.087	Christiana,	60	29.866
Madeira,	32 30	30.126	Bergen,	60	29.703
Tripoli,	33	30.213	Hardanger,	60	29.700
Palermo,	38	30.036	Reikiavig,	64	29.607
Naples,	41	30.012	Godthaab,	64	29.603
Florence,	43 30	29.996	Eyafiord,	66	29.669
Avignon,	44	30.000	Godhaven,	69	29.674
Bologna,	44 30	30.008	Upernavik,	73	29.732
Padua,	45	30.008	Mellville Isle,	74 30	29.807
Paris,	49	29.976	Spitzbergen,	75 30	29.795

42. From the preceding tables, it is seen that the barometrical pressure is much less, especially in the southern hemisphere towards the poles than at the equator, although the density towards the poles is much greater, and hence the depression there must be considerable.

ON THE SOLUTION OF EQUATIONS.

By John Borden, Chicago, Illinois.

The general equation of the second degree, $x^2 + Ax + B = 0$, may be solved as follows. Assume $x = -a + b$, $x = -a - b$ as the values of the roots. Then

$$(x + a - b)(x + a + b) = x^2 + 2ax + a^2 - b^2 = x^2 + Ax + B;$$

and therefore $A = 2a$, $B = a^2 - b^2$; or, $a = \frac{A}{2}$, $b = \pm \frac{1}{2}\sqrt{A^2 - 4B}$.

Therefore, $x = -\frac{A}{2} \pm \frac{1}{2}\sqrt{A^2 - 4B}$, as by the usual method.

The general form of the cubic equation is

$$(1) \qquad x^3 + Ax^2 + Bx + C = 0.$$

If we suppose one root known, as $x = -c$, then the quotient obtained by dividing (1) by $x + c$ must equal zero; that is,

$$x^2 + (A - c)x - (A - c)c + B = 0.$$

Therefore the other two roots are

$$(2) \qquad x = -\frac{A - c}{2} \pm \sqrt{A^2 + 2Ac - 4B - 3c^2}.$$

By making $x = x' - \frac{A}{3}$, as in the common transformation, (1) becomes

$$(3) \qquad x'^3 + \left(B - \frac{A^2}{3}\right)x' + C - \frac{AB}{3} + \frac{2A^3}{27} = 0.$$

If, in this equation, $B - \frac{A^2}{3} = 0$, then

(4) $x' = \left(\frac{A^3}{27} - C\right)^{\frac{1}{3}}, \quad x' = -\frac{1}{2}\left(\frac{A^3}{27} - C\right)^{\frac{1}{3}} \pm \frac{1}{2}\left(\frac{A^3}{27} - C\right)^{\frac{1}{3}}\sqrt{-3},$

If the third term of (3) reduces to zero, then

$$x' - 0, \quad x' = \pm\sqrt{\frac{A^2}{3} - B};$$

or lastly, if the third term, as well as the coefficient of x', is zero, then all the values of x' are zero, and the values of x in (1) all equal $-\frac{A}{3}$. But, none of these cases occurring, (3) is of the general form of (1), if $A = 0$; hence if one root $= -c$, the expression for the other two becomes

(5) $$x' = \frac{c}{2} \pm \sqrt{-4B' - 3c^2}.$$

Therefore, the general form of the roots of (3) are

(6) $$x' = -c, \quad x' = \frac{c}{2} + d, \quad x' = \frac{c}{2} - d;$$

and those of (1), since $x = x' - \frac{A}{3}$, are

(7) $$x = -\frac{A}{3} - c, \quad x = -\frac{A}{3} + \frac{c}{2} + d, \quad x = -\frac{A}{3} + \frac{c}{2} - d.$$

The difficulty, which arises in obtaining the values of c and d in terms of the coefficients appears to come from the fact, that any thing which can be predicated as true of one of the roots in general terms is also true of all the others.

There is one other transformation, of which an equation of the form $x^3 + Bx + C = 0$, for instance, is susceptible. If $x' = xy$, then $x'^3 + Bx'y^2 + Cy^3 = 0$, and we may assume $By^2 = Cy^3$; or $By^2 = E$; or $Cy^3 = d$; or, lastly, $x' = x, y = 1$, in which case $y = \frac{1}{x}$, and the equation becomes $y^3 + \frac{B}{C}y^2 + \frac{1}{C} = 0$, in which the roots, or values of y, are the reciprocals of the values of x, and the solution of one equation involves the solution of the other.

By combining equations (6) there results

$$(8) \qquad x^3 - \left(\frac{3\,c^2}{4} + d^2\right)x + \frac{c^3}{4} - c\,d^2 = 0.$$

From this it appears, that, unless d is imaginary, and $d^2 > \frac{3\,c^2}{4}$, the coefficient of x is essentially negative; and as the third term is the product of all the roots with their signs changed, and the two minor roots are of the same sign, and the maximum root of the contrary sign, that the third term and the maximum root will have contrary signs. If the coefficients of (8) be equated with those of

$$(9) \qquad x^3 + B\,x + C = 0,$$

and c be eliminated, the resulting equation is

$$(10) \qquad d^6 + \tfrac{3}{2}\,B\,d^4 + \tfrac{9}{16}\,B^2\,d^2 + \tfrac{1}{16}\,B^3 + \tfrac{27}{64}\,C^2 = 0; \text{ or}$$

$$(11) \qquad d^3 + \tfrac{3}{4}\,B\,d \pm \sqrt{-\tfrac{1}{16}\,B^3 - \tfrac{27}{64}\,C^2} = 0.$$

If the third term of (11) be equal to zero, then $d = 0$, and two of the roots of (8) are equal, as will appear from (6). But if $d = 0$, then the third term of (11) equals zero, and expresses the relative values of B and C in such case; namely, $4\,B^3 = -27\,C^2$. It also follows, as the third term of (8) equals the third term of (9), that $c = (-4\,C)^{\frac{1}{3}}$. And this is the value of the maximum root when the two minor roots are equal. Further, equation (10) shows that d may have six values, or two for each value of c in (6). And upon investigation, it will be found that d equals one half the algebraic difference of the roots of (9), taken two in a set. And since the two minor roots of (9) have the same sign, the values of d are equal one half their arithmetical difference, and at the same time the maximum root of (9) is equal to their arithmetical sum.

By reference to equation (10), it will appear, that, although it is of the sixth degree, yet in form it is of the third; and is composed of the two cognate factors represented in (11). And this leads to a consideration of the cognate factors into which any equation may be

decomposed. For, suppose the general cubic equation to be put under the form

$$(12)\quad (x^2 - a^2)(x^2 - b^2)(x^2 - c^2) = x^6 - A' x^4 + B' x^2 - C' = 0,$$

and assume

$$(x^3 + A x^2 + B x + C)(x^3 - A x^2 + B x - C) = 0$$

as two of the cubic factors into which it may be decomposed. By multiplying we obtain

$$(13)\quad \begin{aligned} &x^6 - (A^2 - 2B) x^4 + (B^2 - 2AC) x^2 - C^2 = 0 = (12) \\ &x^6 - \qquad\quad A' x^4 + \qquad\quad B' x^2 - C' = 0; \end{aligned}$$

and by equating the coefficients we have

$$(14)\qquad A^2 - 2B = A', \quad B^2 - 2AC = B', \quad C^2 = C';$$

and by eliminating to find A we get

$$(15)\qquad A^4 - 2A'A^2 - 8A\sqrt{C'} + A'^2 - 4B' = 0,$$

which is an equation of the fourth degree, wanting its second term. But (12) is of the form

$$(x^2 - a^2)(x^2 - b^2)(x^2 - c^2) = 0,$$

and equations (13) are respectively of the form

$$(x + a)(x + b)(x + c) = 0, \quad (x - a)(x - b)(x - c) = 0.$$

Therefore, A is equal to the sum of the square roots of the roots of (12), considered as an equation of the third degree. Therefore, the roots of an equation of the form of (15) are determined; and the bi-quadratics are solved. Or, by eliminating A from equations (14), we obtain

$$(16)\qquad B^4 - 2B'B^2 - 8C'B + B'^2 - 4A'C' = 0,$$

which is also an equation of the fourth degree, wanting its second term, and this furnishes another solution of bi-quadratics. For B is equal to the sum of the products of the square roots of the roots of

(12) combined, two in a set, (12) being taken as an equation of the third degree.

By examining the cognate quadratic factors of an equation of the second degree, a relation of the like kind is established between equations of the second and third degrees, as follows:

(17) $$(x^3-a^3)(x^3-b^3)=0=x^6-A'x^3+B',$$

(18) $(x-a)(x-b)=0=x^2-Ax+B$, the factor sought.

$$(x^3-a^3)=(x-a)(x+\tfrac{a}{2}\pm\tfrac{a}{2}\sqrt{-3})\,;\ x^3-b^3=(x-b)(x+\tfrac{b}{2}\pm\tfrac{b}{2}\sqrt{-3}).$$

(19) $$\left(x+\frac{a}{2}+\frac{a}{2}\sqrt{-3}\right)\left(x+\frac{b}{2}+\frac{b}{2}\sqrt{-3}\right)$$
$$=x^2+\left(\frac{A}{2}+\frac{A}{2}\sqrt{-3}\right)x-\frac{B}{2}+\frac{B}{2}\sqrt{-3}.$$

(20) $$\left(x+\frac{a}{2}-\frac{a}{2}\sqrt{-3}\right)\left(x+\frac{b}{2}-\frac{b}{2}\sqrt{-3}\right)$$
$$=x^2+\left(\frac{A}{2}-\frac{A}{2}\sqrt{-3}\right)x-\frac{B}{2}-\frac{B}{2}\sqrt{-3}.$$

But the right-hand members of equations (18), (19), (20) multiplied together, give (17).

$$\therefore\quad x^6-(A^3-3AB)x^3+B^3=x^6-A'x^3+B'=0.$$

(21) Whence $A^3-3AB=A'$, $B^3=B'$, in which $a^3+b^3=A'$, $a^3b^3=B'$, and $a+b=A$, $ab=B$. Whence

(22) $$A=\left(\frac{A'}{2}+\tfrac{1}{2}\sqrt{A'-4B'}\right)^{\frac{1}{3}}+\left(\frac{A'}{2}-\tfrac{1}{2}\sqrt{A'-4B'}\right)^{\frac{1}{3}}.$$

Therefore, if $x^3+px=q$ be identical with (21), then $B'=\frac{1}{27}p^3$, $A'=q$, and

$$x=\left(\frac{q}{2}+\sqrt{\frac{q^2}{4}-\frac{p^3}{27}}\right)^{\frac{1}{3}}+\left(\frac{q}{2}-\sqrt{\frac{q^2}{4}-\frac{p^3}{27}}\right)^{\frac{1}{3}}.$$

which is the well known Cardan formula.

If in (17), (x^3+a^3) (x^3+b^3) had been taken, the same result would have been obtained.

If in (17), $(x^3+a^3)(x^3-b^3)$ had been taken, then

$$(23) \qquad A^3+3AB=A'$$

would result, which corresponds to $x^3-px=q$, and

$$(24) \qquad x=\left(\frac{q}{2}+\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}\right)^{\frac{1}{3}}+\left(\frac{q}{2}-\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}\right)^{\frac{1}{3}}.$$

Equations (22) and (24) are the same as $A=a+b$. And from the values of a and b, the values of $\frac{a}{2}\pm\frac{a}{2}\sqrt{-3}$, $\frac{b}{2}\pm\frac{b}{2}\sqrt{-3}$ can be obtained; also, from the various combinations of the six values which all enter into (17), its different quadratic factors might be constructed. If B' in that equation be supposed to arise from $B.B.B$, then A must correspond to such a combination of the values. Now in such case the combination is

$$(-a)(-b);\ \left(\frac{a}{2}+\frac{a}{2}\sqrt{-3}\right)\left(\frac{b}{2}-\frac{b}{2}\sqrt{-3}\right);$$
$$\left(\frac{a}{2}-\frac{a}{2}\sqrt{-3}\right)\left(\frac{b}{2}+\frac{b}{2}\sqrt{-3}\right).$$

And from such a combination the corresponding values of A can be constructed; and these are the different values of A or x in (21) and (23), and correspond with the CARDAN formulas. By comparing the values of A as thus constructed with the forms as given in (5), we find, by reduction, that the roots of an equation of the form $x^3+Bx+C=0$, (25) are

$$(26) \quad x=(a+b),\ x=-\frac{a+b}{2}+\frac{a-b}{2}\sqrt{-3},\ x=-\frac{a+b}{2}-\frac{a-b}{2}\sqrt{-3},$$

in which

$$a=\left(-\frac{C}{2}+\sqrt{\frac{C^2}{2}+\frac{B^3}{27}}\right)^{\frac{1}{3}},\quad b=\left(-\frac{C}{2}-\sqrt{\frac{C^2}{4}+\frac{B^3}{27}}\right)^{\frac{1}{3}}.$$

If the roots are all real, it is evident that $a-b$ must be imaginary. But this occurs when B is negative, and $\frac{B^3}{27}>\frac{C^2}{4}$, and the values of a and b are not then obtainable by a direct reduction, and the equation is then said to belong to the irreducible case. The method of

obtaining the values of a and b in such case, is by a development of their values. If

$$(27) \qquad m = -\frac{4B^3 + 27C^2}{C^2},$$

the value of x in (25) can be put under the form

$$(28) \quad x = (-4C)^{\frac{1}{3}}\left(1 - \frac{1}{9}m^{\frac{1}{2}}\sqrt{-3}\right)^{\frac{1}{3}} + (-4C)^{\frac{1}{3}}\left(1 + \frac{1}{9}m^{\frac{1}{2}}\sqrt{-3}\right)^{\frac{1}{3}},$$

or by developing

$$(29) \quad x = (-4C)^{\frac{1}{3}}\Big(1 + \frac{1}{3^5}m - \frac{10}{3^{11}}m^2 + \frac{154}{3^{17}}m^3 - \frac{935}{3^{22}}m^4 + \frac{11.13.17.23}{3^{29}}m^5 - \frac{8.13.17.23.29}{3^{35}}m^6 + \frac{17.20.23.29.38}{3^{40}}m^7 - \&c.\Big);$$

or,

$$(30) \quad x = (-4C)^{\frac{1}{3}}\Big(1 + \frac{m}{\log^{-1}2.385606} - \frac{m^2}{\log^{-1}4.247141} + \frac{m^3}{\log^{-1}5.921154} - \frac{m^4}{\log^{-1}7.523670} + \frac{m^5}{\log^{-1}9.085424} - \frac{m^6}{\log^{-1}10.622890} + \frac{m^7}{\log^{-1}12.143496} - \&c.\Big).$$

Equations (29) and (30) appear to give the value of the maximum root, and these series are rapidly converging, when m is a small number. If $m = 0$, the two minor roots are equal, and the maximum root equals $(-4C)^{\frac{1}{3}}$. If both terms of the values of x in (28) be developed in the descending powers of the imaginary, the values of x become

$$(31) \quad x = -(-4C)^{\frac{1}{3}}\Big(m^{-\frac{1}{3}} - 5m^{-\frac{4}{3}} + 66\,m^{-\frac{7}{3}} - 17.66m^{-\frac{10}{3}} + 17.23.55\,m^{-\frac{13}{3}} - 17.23.29.39m^{-\frac{16}{3}} + 17.23.24.29.35m^{-\frac{19}{3}} - \&c.\Big).$$

This equation appears to give the minimum root, and will be converging when m is a large number. If one term of the value of x in (28) be dropped in the ascending powers of the imaginary, and the other in the descending, then the terms involving the imaginary cannot cancel each other, for the sum of the real terms in that case equals the half sum of (29) and (30), which is not a root of (25). Nor are we to suppose that the sum and difference of any number

of imaginary terms are equal to a real number. Therefore the expression for the value of x in this case is part real and part imaginary. But in the irreducible case now under consideration, all the roots are real; therefore this value of x is not a root of (25), but is still to be found among the combinations of the six roots or values which go to make up the quadratic factor of (17). If the values of x in (26) be combined, there results

$$(32) \qquad x^3 - 3\,a\,b\,x - a^3 - b^3 = 0,$$

from which, by comparing with $x^3 + B\,x + C = 0$, we have

$$(33) \qquad -3\,a\,b = B, \text{ or } -27\,a^3\,b^3 = B^3\,;\ -a^3 - b^3 = C.$$

But it is evident, that, if $-\frac{a}{2} \pm \frac{a}{2}\sqrt{-3}$, $-\frac{b}{2} \pm \frac{b}{2}\sqrt{-3}$ were substituted in the first set of (33), $-a^3 - b^3 = C$ would be the same. The same would be the case if $-\frac{B}{2} \pm \frac{B}{2}\sqrt{-3}$ were substituted. Therefore the values of a and b derived therefrom involve these various cases, and the equations

$$(34) \qquad x^3 + B\,x + C = 0,\ x^3 + \left(-\frac{B}{2} + \frac{B}{2}\sqrt{-3}\right)x + C = 0,$$
$$x^3 + \left(-\frac{B}{2} - \frac{B}{2}\sqrt{-3}\right) + C = 0$$

are all equally solved by the Cardan formulas.

It results from these developments, that an expression of the form $(m+n)^{\frac{1}{3}}$ does not give always the same value, when developed according to the ascending powers of n, as when developed according to the ascending powers of m.

If an equation of the fourth degree be examined with reference to its quadratic factors; as, for instance, if

$$(35) \qquad (x^2 + Ax + B)(x^2 + Cx + D) = x^4 + ax^3 + bx^2 + cx + d = 0,$$

and the four roots are a', b', c', d', it is plain that B, C, D, or A will have six values. For the values of B are $a'b'$, $a'c'$, $a'd'$, $b'c'$, $b'd'$, $c'd'$.

Thence the value of any one of the coefficients of either of the quadratic factors, when obtained, should be expressed in the terms of an equation of the sixth degree. By equating, we get

(36) $A + C = a,\ A\,C + B + D = b,\ B\,C + A\,D = c,\ B\,D = d.$

By eliminating to find the value of B, we have, after reduction,

$$(37)\quad \begin{aligned} B^6 - b\,B^5 + (a\,c - d)\,B^4 + (2\,b\,d - c^2 - a^2\,d)\,B^3 \\ + (a\,c\,d - d^2)\,B^2 - b\,d^2\,B + d^3 = 0. \end{aligned}$$

Hence, assuming

$$(38)\quad x^6 + f\,x^5 + g\,x^4 + h\,x^3 + (g\,l^{\frac{1}{3}} - 2\,l^{\frac{2}{3}})\,x^2 + f\,l^{\frac{2}{3}}\,x + l = 0,$$

and equating the coefficients of (37) and (38), the values of a, b, c, d, the coefficients of (35), can easily be obtained. Hence the roots of an equation of the sixth degree of the form (38), as also its reciprocal, with their various modifications, are dependent upon the roots of the general bi-quadratic (35); for the roots of (38) are the roots of (35) multiplied together, two in a set.

This method of examining an equation by its factors, furnishes a general method of solution of equations; the only objection being, that, as we reason from the less degree, we are bound to accept the greater degree, in whatever form the equation may present itself. And if any equation be taken as the subject of examination, the question put is, What are the equations of the higher degrees, whose forms depend for solution upon the equation under examination? It may be said, that there are certain forms of equations of all degrees above the second, which may be evolved from equations of the second degree, and their solution thereby obtained. As, for instance, the equation of the fifth degree of the form

$$(39)\qquad x^5 + B\,x^3 + \frac{B}{5}\,x + C = 0$$

has for the value of x,

$$(40)\quad x = \left(-\frac{C}{2} + \sqrt{\frac{C^2}{4} + \left(\frac{B}{5}\right)^5}\right)^{\frac{1}{5}} + \left(-\frac{C}{2} - \sqrt{\frac{C^2}{4} + \left(\frac{B}{5}\right)^5}\right)^{\frac{1}{5}};$$

and its reciprocal,

$$(41) \qquad x^5 + \frac{B^2}{56} x^4 + \frac{B}{C} x^2 + \frac{1}{C} = 0,$$

is therefore solved also.

Also, the form of the equation of the seventh degree is

$$(42) \qquad x^7 + B x^5 + \tfrac{2}{7} B^2 x^3 + \tfrac{1}{49} B^3 x + C = 0,$$

its solution is

$$(43) \quad x = \left(-\frac{C}{2} + \sqrt{\frac{C^2}{4} + \left(\frac{B}{7}\right)^7}\right)^{\frac{1}{7}} + \left(-\frac{C}{2} - \sqrt{\frac{C^2}{4} + \left(\frac{B}{8}\right)^7}\right)^{\frac{1}{7}};$$

its reciprocal is

$$(44) \qquad x^7 + \frac{B^3}{C} x^6 + \tfrac{2}{7} B^2 x^4 + \frac{B}{49\,C} x^2 + \frac{1}{C} = 0.$$

In all these cases, there are only two arbitrary quantities, B and C, from which to determine the coefficients; but by the introduction of the transformation $x' = y\,x$, the coefficients may be variously modified. The square root of an equation of the fifth degree of the general form will probably evolve an equation of the eighth degree; and if so, it will lack but one arbitrary quantity, with the assistance of the transformations, of showing the dependence of the general equation of the eighth degree upon one of the fifth.

If an equation of the third degree of the form

$$(45) \qquad x^{15} - A' x^{10} + B' x^5 - C' = 0$$

be resolved into its five factors, by the method as above, we obtain

$$(46) \qquad A^5 - 5\,B\,A^3 + 5\,C A^2 + 5\,B^2 A - 5\,B\,C = A';$$

$$(47) \qquad B^5 - 5\,A\,C\,B^3 + 5\,C^2 B^2 + 5\,A^2 C^2 B - 5\,A\,C^3 = B';$$

$$(48) \qquad C^5 = C',$$

from which it will require some industry to eliminate A, B, C.

There is but one other matter to be named; and that is as to the question, How are the imaginary roots to be interpreted? In the quadratic equation, when the roots become imaginary, the conditions are said to be impossible, and they are used to detect such

impossible conditions. But in equations of a degree above the second, they may be part real and part imaginary; and in such case, how is the test to be used? For in such case, the imaginary roots are as truly roots of the equation as the real roots. There is but little doubt that they admit of geometrical interpretation. Otherwise, equations of the higher orders, instead of being of extraordinary power, from their extreme flexibility and great capacity, may be said to be labyrinths filled with nests of ghostly quantities.

Editorial Items.

The following gentlemen have sent us solutions of the Prize Problems in the May number of the Monthly: —

C. M. Woodward, Junior Class, Harvard College, answered all the questions. (Benjamin Peirce, Prof.)

Horace Otis, Adams Centre, N. Y., answered I. and IV.

P. Barton, Amsterdam, N. Y., answered all but III.

Roland Thompson, Junior Class, Jefferson College, Cannonsburg, Penn., answered I. and IV. (John Fraser, Prof.)

David Trowbridge, Perry City, Schuyler Co., N. Y., answered all the questions.

W. F. Osborne, Sophomore Class, Wesleyan University, Middletown, Conn., answered all the questions. (J. M. VanVleck, Prof.)

Asher B. Evans, Madison University, Hamilton, N. Y., answered all the questions.

Gustavus Frankenstein, Springfield, Ohio, answered all the questions.

George A. Osborne, Jr., Lawrence Scientific School, answered all the questions. (H. L. Eustis, Prof.)

George W. Jones, Jr., Senior Class, Yale College, answered all the questions. (H. A. Newton, Prof.)

W. Murray Sterling, Student, Baltimore, Md., answered questions I., II. and IV. (Timothy Crimmin, Teacher.)

A Student, High School, Baltimore, Md., answered questions I., II. and IV. (James McIntire, Prof.)

It gives us pleasure to add the following names to our list of coöperators and contributors: Edwin Haas, Burlington, N. J.; R. C. Matthewson, San Francisco, Cal.; W. C.

DENNIS, Key West, Florida; AUGUST SONNTAG, Acting Assistant at the Dudley Observatory, Albany, N. Y.; WILLIAM J. LEWIS, San Francisco, Cal. . . . . To the following Card, which we are permitted to lay before the readers and friends of the MATHEMATICAL MONTHLY, we have nothing to add, except that it amply repays us for the care and labor we have already devoted to the work. All we ask, is such a support as shall enable us to publish promptly all we receive worthy of being put in more permanent form: —

"*Boston*, May 25th, 1859.

"The undersigned, having watched with great interest the establishment of a new MATHEMATICAL JOURNAL at Cambridge, under the editorship of Mr. J. D. RUNKLE, are desirous of calling the attention of the patrons of sound learning to this work. The 'MATHEMATICAL MONTHLY' commenced its existence in October, 1858, and has been conducted on a plan that cannot fail to make it an instrument of great good. It is addressed to students as well as to professors; and has doubtless already given a new impulse to mathematical studies, wherever it has been introduced. A work of this kind should not be suffered simply to live. It has now about eleven hundred subscribers; and is not, perhaps, likely to be altogether discontinued, while its present support remains. But its friends should not be satisfied with this. A liberal subscription should insure to it a vigorous, energetic, long-continued life; and should enable it not only to preserve its present excellent form, and the stimulus of its prizes,— now amounting to about three hundred dollars,— but to make new improvements; to increase its size, without increasing its terms; to secure the best matter, by paying contributors, if necessary; and to compensate its editor, in part at least, for the time he bestows upon it.

"This appeal is made from no suggestion of the editor, but from the conviction that the work deserves a wide patronage, and must secure it, in order to be *permanently* successful. The circle of strictly mathematical readers in this country is yet small. To enlarge it, nothing can be more surely relied on than a well-conducted Journal; but until this is done, and the work is thus made to support itself, the aid of *all* true friends of Science must be invoked.

"JAMES WALKER.
JARED SPARKS.
BENJAMIN PEIRCE.
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We hope that Prof. WILLIAM RUTHERFORD, of the Royal Military Academy, Woolwich, England, will pardon us for giving, in connection with the above, an extract from his polite note of May 4th, 1859:—

"I have also got all the numbers of your new Monthly Mathematical publication, with which I am very much pleased. It will be a useful work, and I regret we have not any similar publication in Britain."

Prof. GIBBES has sent us the following errata: On page 338, last line, for >0.250, read <0.250; page 339, line —5, in the value of y, put exponent $\frac{1}{3}$ outside the parenthesis. A few unimportant verbal errors we omit, as they will give the reader no difficulty. On page 350, for 63 cents, read 36.

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